SOLUTIONS FOR ADMISSIONS TEST IN
MATHEMATICS, COMPUTER SCIENCE AND JOINT SCHOOLS
WEDNESDAY 31 OCTOBER 2018

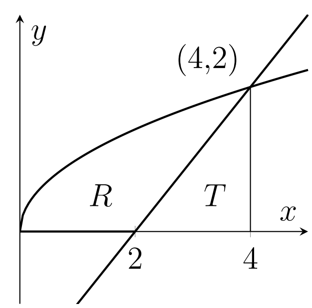
Mark Scheme:

Each part of Question 1 is worth 4 marks which are awarded solely for the correct answer.

Each of Questions 2-7 is worth 15 marks

QUESTION 1:

A. The curves $y = \sqrt{x}$ and $y = x - 2$ intersect when $x = 4$. We want the area of region R in the sketch below. We will integrate $\sqrt{x}$ from 0 to 4 and subtract the area of triangle T. This gives the required area as $\int_0^4 \sqrt{x}\,\mathrm{d}x - \frac{1}{2} \cdot 2 \cdot 2 = \left[\frac{2}{3}x^{3/2}\right]_0^4 - 2 = \frac{10}{3}$. **The answer is (d).**



B. Substitute in $y = e^{kx}$ and use the fact that the derivative of e^{kx} is ke^{kx}. This gives

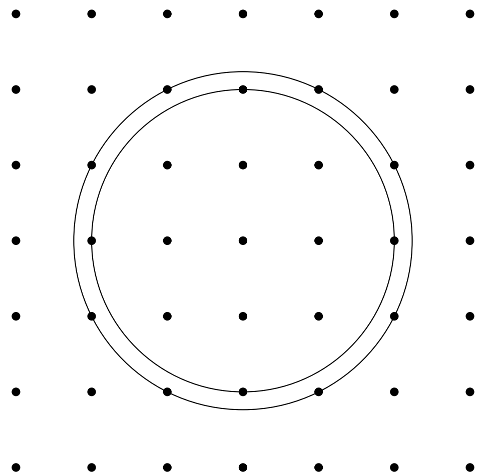
$$\left(k^2e^{kx} + ke^{kx}\right)\left(ke^{kx} - e^{kx}\right) = ke^{2kx}$$

Now e^{kx} is never zero, so we may divide each side by e^{2kx}, giving an cubic equation for k; $k(k+1)(k-1) = k$. This has three solutions; $k = 0$ or $k = \sqrt{2}$ or $k = -\sqrt{2}$. **The answer is (d).**

C. We have $ax^2 + c = bx^2 + d$ if and only if $(a-b)x^2 = (c-d)$. If $(a-b) = 0$ then we either have no solutions (if $c \neq d$) or infinitely many solutions (if $c = d$). Otherwise, we can safely divide by $(a-b)$ and get $x^2 = (c-d)/(a-b)$. Then there are two roots precisely when $(c-d)/(a-b) > 0$. Multiplying up by $(a-b)^2$, which is positive, our expression becomes $(c-d)(a-b) > 0$. **The answer is (e).**

D. We can complete the square for $f(x) = \left(x - \frac{5}{2}\right)^2 + \frac{3}{4}$. This has a minimum at $\left(\frac{5}{2}, \frac{3}{4}\right)$. Then the graph of $f(x-2)$ is a translation of the graph of $f(x)$ by two in the positive x-direction. So the minimum is translated to $\left(\frac{9}{2}, \frac{3}{4}\right)$. **The answer is (b).**

E. The circle of radius 2 contains 13 points (the point at $(1,1)$ is inside, but the point at $(1,2)$ is outside, as $\sqrt{2} < 2 < \sqrt{5}$). So $n = 13$ and $2n - 5 = 21$. We want the smallest circle that contains 21 points, which is eight more than the previous circle. The next-closest points to the origin are the points at $(\pm1, \pm2)$ and $(\pm2, \pm1)$ of which there are eight. So the radius must increase by $\sqrt{5} - 2$. **The answer is (a).**



F. The particle's motion is restricted to the wedge bounded by the lines $y = 2x$ and $y = x/2$, and the particle could reach any grid point on the boundary on the wedge just by using a single repeated move. Since the point $(25, 75)$ is outside this wedge, we want the closest point on the boundary of the wedge to this point. We must minimise $(x-25)^2+(2x-75)^2$, which occurs when $2x-50+8x-4\cdot 75 = 0$, that is when $x = 35$. The distance from $(35, 70)$ to $(25, 75)$ is $\sqrt{10^2+5^2} = 5\sqrt{5}$. **The answer is (b).**

G. The curves meet and have the same gradient, so

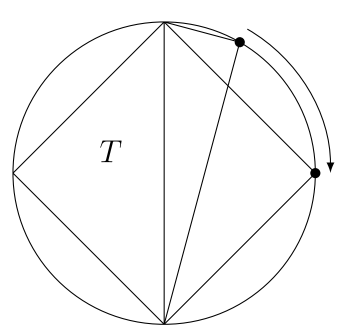
$$\begin{aligned} x^2 + c =& x^{1/2} \\ 2x =& \frac{1}{2x^{1/2}} \end{aligned}$$

The second equation gives $x = 4^{-2/3}$, and then substituting this into the first equation gives

$$c = 4^{-1/3} - 4^{-4/3} = 4^{-1/3}\left(1-\frac{1}{4}\right) = \frac{3}{4\sqrt[3]{4}}$$

The answer is (b).

H. The triangle T could be any size; for instance it could be a right-angled triangle in a circle of arbitrary radius. Given such a triangle T, the triangle S could be any size smaller than T; we can change the size of triangle S from zero to t by moving the other vertex of triangle T around the circle.



So we have $s < t$, but otherwise s and t could be any positive numbers. We want to minimise

$$\frac{4s^2+t^2}{5st} = \frac{1}{5}\left(4\left(\frac{s}{t}\right)+\left(\frac{s}{t}\right)^{-1}\right).$$

Write $u = s/t$, so that $0 < u < 1$. The expression $4u + u^{-1}$ is minimised when $4 - u^{-2} = 0$, which is when $u = 1/2$ (which is between 0 and 1). The value at $s/t = 1/2$ is $4/5$. **The answer is (c).**

I. Note that $x^8 + 4x^6y + 6x^4y^2 + 4x^4y^3 + y^4 = (x^2 + y)^4$. So the equation is $(x^2 + y)^8 = 1$, which has solutions when $y = 1 - x^2$ or when $y = -1 - x^2$. This is a pair of parabolas, and y is negative for large x in either case. **The answer is (c).**

J. If (x, y) is a solution to this equation, then so is (y, x). So the graph must have reflectional symmetry in the line $x = y$. So A cannot be a sketch of this curve. **The answer is (c).**

As a check, note that C could be a sketch with $p(x) = x^2 - 1/2$ and D could be a sketch with $p(x) = x$. For B, look at the part of the sketch with y constant. Along this line segment, $p(x) + p(y) = 0$ and $p(y)$ is a constant. But if this polynomial is zero in some interval, it must be zero for all x, so $p(x)$ must be constant! But then either all points should be in the sketch (if the constant is zero) or none should (otherwise). This sketch is neither, and therefore impossible.

2.

(i) We see

$$TS(x,y) = (-y, x+1), \qquad ST(x,y) = (-y+1, x)$$

aren't equal.

1 mark

(ii) We see

$$T^2(x,y) = (-x,-y), \qquad T^3(x,y) = (y,-x), \qquad T^4(x,y) = (x,y).$$

2 marks

So if $T^n(x,y) = (x,y)$ for all x, y then n is a multiple of 4.

1 mark

(iii) Note that (for example)

$$T^2ST^2(x,y) = T^2S(-x,-y) = T^2(-x+1,-y) = (x-1,y).$$

2 marks

To achieve this we cannot use S alone, and if T is involved then it must be applied at least four times, so that the xy-plane has regained its original orientation. We can't use T alone, since that just gives the transformations above. The minimum is five. **2 marks**

(iv) Using S and (iii) we know we can horizontally translate left or right, so we can move $(0,0)$ to $(b,0)$. Then T takes it to $(0,b)$. Then horizontal translation takes it to (a,b).

3 marks

(v) The points on the original curve are given by (x, x^2+2x+2). Applying S to these points gives $(x+1, x^2+2x+2)$. If we write (u,v) for the new coordinates, then $u = x+1$ and $v = x^2+2x+2$. So the equation of the new curve is $v = (u-1)^2 + 2(u-1) + 2$.

[Alternative method: Note that C has equation $y-1 = (x+1)^2$, a parabola with vertex $(-1,1)$. Translating C to the right (i.e. applying S) gives the curve $y-1 = x^2$.]

[Alternative method: $(x,y) \in S(C)$ iff $S^{-1}(x,y) = (x-1,y) \in C$ iff $y-1 = x^2$.]

2 marks

Applying T to the points of the parabola gives the curve $(-x^2-2x-2, x)$. If we write (u,v) for the new coordinates, then $u = -x^2-2x-2$ and $v = x$. So the equation of the new curve is $u = -v^2 - 2v - 2$.

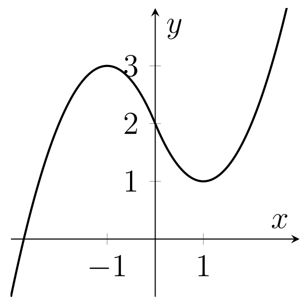
[Alternative method: rotating C by a right angle anti-clockwise, (i.e. applying T) that vertex is mapped to $(-1,-1)$. By considering the corresponding distances that gave C's original equation we see the image has equation $-(x+1) = (y+1)^2$ or $x + y^2 + 2y + 2 = 0$.]

[Alternative method: $(x,y) \in T(C)$ iff $T^{-1}(x,y) = (y,-x) \in C$ iff $-x-1 = (y+1)^2$.]

2 marks

3.

(i) Looking for extrema, join, and general shape.



2 marks

(ii) Since $g(0) = 2$ we have

$$m(x) = \frac{(x-1)^2 + 1 - 2}{x} = x - 2 \text{ if } x > 0,$$

and

$$m(x) = \frac{3 - (x+1)^2 - 2}{x} = -2 - x \text{ if } x < 0.$$

1 mark

(iii) For $x > 0$, $m(x) + 2 = (x - 2) + 2 = x$. For $x < 0$, $m(x) + 2 = -x$.

1 mark

(iv) If we take the limit $x \to 0$, then the gradient of the line tends to the gradient of the function. For $x > 0$, $m(x) + 2 = x$. This tends to zero, so $m(x)$ tends to -2.

1 mark

(v) h' is a quadratic with zeroes at p and q, so $h'(x) = A(x-p)(x-q)$ for some A. Then $h''(x) = A(2x - p - q)$ and in particular $h''(p) = A(p - q)$. We know that h has a maximum at p, $h''(p) \le 0$, and we know that $p - q < 0$. Therefore $A \ge 0$. Now for $p < x < q$ we have $(x - p)(x - q) < 0$ and so $h'(x) = A(x - p)(x - q) \le 0$.

4 marks

(vi) By the previous part, $h'(0) < 0$ and so $c > 0$.

1 mark

We know that $h'(x)$ is a quadratic with roots at ± 1. So $h'(x) = A(x^2 - 1)$ for some A. Moreover, $h'(0) = -3c$ so $A = 3c$. Now integrate for $h(x) = 3c(x^3/3 - x) + d$ where the constant follows from $h(0) = d$.

3 marks

(vii) If the graphs are the same then $g'(0) = h'(0)$ so $c = \frac{2}{3}$, and $g(0) = h(0)$ so $d = 2$, and $h(x) = \frac{2}{3}x^3 - 2x + 2$. But then $h(1) = \frac{2}{3} \neq 1 = g(1)$.

2 marks

4.

(i) By the cosine rule

$$6 = 4 + (\sqrt{3}-1)^2 - 4(\sqrt{3}-1)\cos\angle CBA$$

which rearranges to give $\cos\angle CBA = -1/2$ and so $\angle CBA = 120°$.

2 marks

Since this angle is obtuse, A and B lie on the same side of CD.

2 marks

(ii) Half the length of CD is $(\sqrt{3}-1)\cos 60° = (\sqrt{3}-1)\sqrt{3}/2$. So the total length is $3-\sqrt{3}$.

2 marks

By the cosine rule on triangle CAD, we have

$$(3-\sqrt{3})^2 = 6 + 6 - 12\cos\angle CAD$$

and so $\cos\angle CAD = \sqrt{3}/2$ and $\angle CAD = 30°$.

2 marks

(iii) The segment delimited by the chord CD and the circle S_2 has area

$$(\sqrt{3}-1)^2\left(\frac{\pi}{3}-\frac{\sqrt{3}}{4}\right) = (4-2\sqrt{3})\left(\frac{\pi}{3}-\frac{\sqrt{3}}{4}\right).$$

The segment delimited by the chord CD and the circle S_1 has area

$$(\sqrt{6})^2\left(\frac{\pi}{12}-\frac{1}{4}\right) = 6\left(\frac{\pi}{12}-\frac{1}{4}\right).$$

The area of the shaded region is therefore

$$(4-2\sqrt{3})\left(\frac{\pi}{3}-\frac{\sqrt{3}}{4}\right) - 6\left(\frac{\pi}{12}-\frac{1}{4}\right) = \frac{\pi}{6}(5-4\sqrt{3}) + 3 - \sqrt{3}.$$

3 marks

(iv) The area of the larger segment created by CE and S_1 must equal half the area of the region covered by S_1 and S_2. Thus

$$\begin{aligned}(\sqrt{6})^2\left(\frac{360 - x + \sin x}{360}\right) &= \frac{1}{2}\,(\text{area of } S_1 + \text{area of shaded region})\\ &= \frac{1}{2}\left(6\pi + \frac{\pi}{6}(5-4\sqrt{3}) + 3 - \sqrt{3}\right)\\ &= \frac{1}{2}\left(\frac{\pi}{6}(41-4\sqrt{3}) + 3 - \sqrt{3}\right).\end{aligned}$$

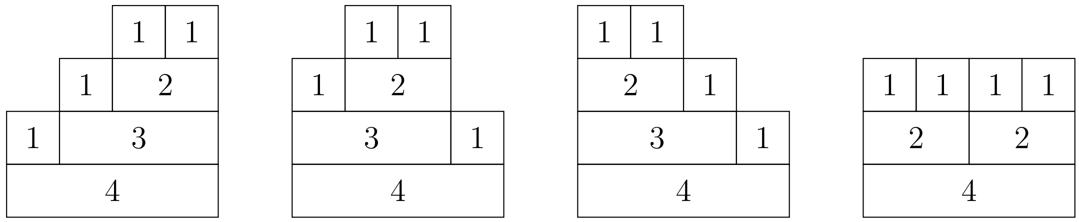
2 marks

Now $0 < x < 180°$ and in this interval, as x increases, the area of the larger segment created by a pie of angle x is strictly increasing. Differently said, the left hand side is a strictly decreasing function of x. (It is acceptable if the candidate argues using derivative.) Hence the above equation has a unique solution.

2 marks

5.

(i)



2 marks

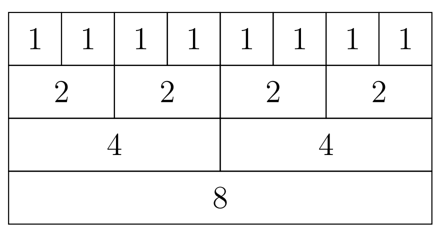
(ii) It is n. There is an n-tower of height n: take an n-brick, stack an $(n-1)$-brick and a 1-brick on top of it, stack an $(n-2)$-brick and a 1-brick on top of the $(n-1)$-brick, $\ldots$, stack two 1-bricks on top of the 2-brick. There is no higher n-tower, as on each level the maximal width of any brick decreases by at least 1.

3 marks

(iii) It follows from the argument in the previous part that all n-towers of maximal height consist of the same number of the same bricks, namely one k-brick for every $k \leq n$ and $n-1$ additional 1-bricks. So the answer is $(\sum_{k=1}^{n} k) + (n-1) \cdot 1 = \frac{n(n+1)}{2} + n - 1 = \frac{n^2}{2} + \frac{3n}{2} - 1$.

2 marks

(iv) Generalize the 4-tower drawn last in the solution of part (i) for all powers of 2. For instance, here is an 8-tower:



Continuing this pattern, there is a 2^k-tower of height $k+1$ for all $k \geq 1$.

3 marks

(v) $t_n = \displaystyle\sum_{k=1}^{n-1} t_k t_{n-k}$.

$t_2 = t_1 \cdot t_1 = 1, \qquad t_3 = t_1 \cdot t_2 + t_2 \cdot t_1 = 2, \qquad t_4 = t_1 \cdot t_3 + t_2 \cdot t_2 + t_3 \cdot t_1 = 5,$

$t_5 = t_1 \cdot t_4 + t_2 \cdot t_3 + t_3 \cdot t_2 + t_4 \cdot t_1 = 14, \qquad t_6 = t_1 \cdot t_5 + t_2 \cdot t_4 + t_3 \cdot t_3 + t_4 \cdot t_2 + t_5 \cdot t_1 = 42.$

3 marks

(vi) By the previous part we have:

$$t_{2n} = \sum_{k=1}^{2n-1} t_k t_{2n-k} = t_n t_n + 2 \sum_{k=1}^{n-1} t_k t_{2n-k}$$

If t_n is odd then $t_n t_n$ is odd. Hence t_{2n} is odd.
If t_n is even then $t_n t_n$ is even. Hence t_{2n} is even.

2 marks

6.

(i) $\frac{2}{3} = \frac{1}{2} + \frac{1}{6} = \frac{1}{3} + \frac{1}{4} + \frac{1}{12}$;
$\frac{2}{5} = \frac{1}{3} + \frac{1}{15} = \frac{1}{4} + \frac{1}{10} + \frac{1}{20} = \frac{1}{5} + \frac{1}{6} + \frac{1}{30}$;
$\frac{23}{40} = \frac{1}{2} + \frac{1}{20} + \frac{1}{40}$.

3 marks

(ii) If m is smallest such that $\frac{1}{m} \leq q$ then $m \geq 2$ and $\frac{1}{m-1} > \frac{a}{b}$. So $a(m-1) < b$, and $am - b < a$, and so

$$\frac{a}{b} - \frac{1}{m} = \frac{am-b}{bm} = \frac{c}{d}$$

satisfies $c < a$.

3 marks

(iii) Given any rational $q = \frac{a}{b}$ with $0 < q < 1$, we can chop off the largest possible reciprocal, and that leaves a remainder $\frac{c}{d}$ with $0 \leq c < a$. Continuing in this way yields a sequence of remainders with decreasing numerators, which must eventually terminate. Note that the reciprocals generated by the process must be distinct, because at each stage $\frac{1}{m} > \frac{q}{2}$, for otherwise $\frac{1}{m-1} \leq \frac{1}{m/2} \leq q$. If $\frac{1}{n}$ is generated later in the process, then

$$\frac{1}{n} \leq q - \frac{1}{m} < \frac{q}{2} < \frac{1}{m},$$

so $n > m$.

4 marks

(iv) We see $\frac{1}{4} < \frac{4}{13} < \frac{1}{3}$, so we write $\frac{4}{13} = \frac{1}{4} + \frac{3}{52}$. Next, $\frac{1}{18} < \frac{3}{52} < \frac{1}{17}$, so we write $\frac{4}{13} = \frac{1}{4} + \frac{1}{18} + \frac{1}{468}$, and we have finished.

2 marks

(v) Given any positive rational q, we can find the last partial sum of $\sum \frac{1}{n}$ that is $\leq q$. The remainder is then less than $\frac{1}{n+1}$, and can be represented using the previous method, generating denominators greater than those taken from the harmonic series.

3 marks

7.

(i) We must reach (i, j) via either $(i-1, j)$ or $(i, j-1)$, and in order to achieve the maximum score at (i, j), we must get to the previous cell by a path that maximises the score. This justifies the relation

$$m(i, j) = c(i, j) + \max\big(m(i-1, j), m(i, j-1)\big).$$

3 marks

(ii) The entries in the table are shown below.

2	2	3	4	4	5	5	5
2	2	3	4	4	4	4	4
2	2	3	3	3	3	4	4
1	2	2	2	3	3	3	3
1	2	2	2	2	2	3	3
0	0	0	2	2	2	3	3
0	0	0	1	1	2	3	3
0	0	0	1	1	2	2	2

2 marks

The maximum score is 5 points.

1 mark

(iii) We must enter cell (i, j) from whichever previous cell has the larger value of m: this will either be the same as $m(i, j)$, or one smaller if there is a coin at (i, j). If both predecessors have an equal score, we can choose arbitrarily between them. By following these directions, we can trace a path from E back to S.

3 marks

One possible path is shown.

2	2	3	4	4	5	5	5
2	2	3	4	4	4	4	4
2	2	3	3	3	3	4	4
1	2	2	2	3	3	3	3
1	2	2	2	2	2	3	3
0	0	0	2	2	2	3	3
0	0	0	1	1	2	3	3
0	0	0	1	1	2	2	2

1 mark

(iv) For each cell, we can tabulate also the number $p(i,j)$ of maximum-achieving paths reaching the cell, setting it to $p(i-1,j)$ or $p(i,j-1)$ according to the predecessor that would be chosen, or to the sum of these if the scores $m(i-1,j)$ and $m(i,j-1)$ are equal.

3 marks

			8	16	24	24	24
		4	8	8	8		
1	2	4	4				
1	1	2					
1	1	1					
1							
1							
1							

So the answer is 24.

2 marks