## SOLUTIONS FOR ADMISSIONS TEST IN
# MATHEMATICS, JOINT SCHOOLS AND COMPUTER SCIENCE
## WEDNESDAY 5 NOVEMBER 2008

**Mark Scheme:**

Each part of Question 1 is worth four marks which are awarded solely for the correct answer.

Each of Questions 2-7 is worth 15 marks

**QUESTION 1:**

**A.** As $y = 2x^3 - 6x^2 + 5x - 7$ then

$$y' = 6x^2 - 12x + 5.$$

The quadratic $y'$ has discriminant $12^2 - 4 \times 6 \times 5 = 24 > 0$ and hence the equation $y' = 0$ has two distinct real roots. **The answer is (c).**

**B.** As $\pi < 10$ then

$$L = \log_{10} \pi < 1.$$

So

$$\sqrt{\log_{10}(\pi^2)} = \sqrt{2L} > \sqrt{L \times L} = L; \qquad \left(\frac{1}{\log_{10} \pi}\right)^3 = L^{-3} > 1; \qquad \frac{1}{\log_{10} \sqrt{\pi}} = \frac{2}{L} > 2.$$

**The answer is (a).**

**C.** We will write $c = \cos\theta$ and $s = \sin\theta$ for ease of notation. Eliminating $y$ from the simultaneous equations

$$cx - sy = 2, \qquad sx + cy = 1;$$

we get

$$2c + s = c\,(cx - sy) + s\,(sx + cy) = \left(c^2 + s^2\right) x = x$$

and similarly eliminating $x$ we find

$$c - 2s = (-s)\,(cx - sy) + c\,(sx + cy) = \left(s^2 + c^2\right) y = y.$$

Hence the equations care solvable for any value of $\theta$. **The answer is (a).**

**D.** By the remainder theorem when a polynomial $p(x)$ is divided by $x - 1$ then the remainder is $p(1)$. So the required remainder here is

$$1 + 3 + 5 + 7 + \cdots + 99 = \frac{50}{2}(1 + 99) = 2500$$

as the series is an arithmetic progression. **The answer is (b).**

**E.** The highest power of $x$ in $(2x^6+7)^3$ is $x^{18}$ and in $(3x^8-12)^4$ is $x^{32}$ so the highest power in $[\cdots]^5$ is $(x^{32})^5=x^{160}$. The highest power of $x$ in $(3x^5-12x^2)^5$ is $x^{25}$ and in $(x^7+6)^4$ is $x^{28}$, so that the highest power of $x$ in $[\ldots]^6$ is $(x^{28})^6=x^{168}$. Thus the highest power of $x$ in $\{\ldots\}^3$ is $(x^{168})^3=x^{504}$. **The answer is (d).**

**F.** Suppose that, when the trapezium rule is used to estimate the integral $\int_0^1 f(x)\,\mathrm{d}x$, an overestimate of $E$ is produced. If the same number of intervals are used in the following calculations then:

(a) to estimate $\int_0^1 2f(x)\,\mathrm{d}x$ an overestimate of $2E$ will be produced, as the relevant graphs have been stretched by a factor of 2 and all areas doubled;

(b) to estimate $\int_0^1 (f(x)-1)\,\mathrm{d}x$ an overestimate of $E$ will be produced, as the relevant graphs have been translated down by 1 and all areas remain the same;

(c) to estimate $\int_1^2 f(x-1)\,\mathrm{d}x$ an overestimate of $E$ will be produced, as the relevant graphs have been translate right by 1 and all areas remain the same;

(d) to estimate $\int_0^1 (1-f(x))\,\mathrm{d}x$ an underestimate of $E$ will be produced, as the relevant graphs have been reflected in the $x$-axis – turning the overestimate to an underestimate – and translated up by 1, which changes nothing with regard to areas. **The answer is (d).**

**G.** As $4x-x^2-5=-(x-2)^2-1$, then $y=(4x-x^2-5)^{-1}$ is always negative and has a minimum value at $x=2$. **The answer is (c).**

**H.** If we set $y=3^x$ then the equation $9^x-3^{x+1}=k$ now reads

$$y^2-3y-k=0.$$

This has solutions

$$y=\frac{3\pm\sqrt{9+4k}}{2}$$

which are real when $k\geqslant -9/4$. As $y=3^x$ then we further need that $y>0$ for $x$ to be real, but this is not a problem as the larger root is clearly positive. **The answer is (a).**

**I.** We have

$$S(1)+S(2)+S(3)+\cdots+S(99)=S(00)+S(01)+\cdots+S(99)$$

and in the 100 two-digit numbers $00,\ldots,99$ there are twenty $0s$, twenty $1s$, $\ldots$, twenty 9s. So

$$S(1)+S(2)+S(3)+\cdots+S(99)=20\times(0+1+\cdots+9)=20\times\frac{10}{2}(0+9)=900$$

and **the answer is (c).**

**J.** Note that

$$(3+\cos x)^2\geqslant(3-1)^2=4;\qquad 4-2\sin^8 x\leqslant 4.$$

So the equation will hold only wehn $\cos x=-1$ and $\sin x=0$. In the range $0\leqslant x<2\pi$ this only occurs at $x=\pi$. **The answer is (b).**

**2.** (i) A fairly obvious pair $(x_1, y_1)$ that satisfy $(x_1)^2 - 2(y_1)^2 = 1$ iis $x_1 = 3$ and $y_1 = 2$.

(ii) Note

$$\begin{aligned} (x_{n+1})^2 - 2(y_{n+1})^2 &= (x_n)^2 - 2(y_n)^2 \\ \iff (3x_n + 4y_n)^2 - 2(ax_n + by_n)^2 &= (x_n)^2 - 2(y_n)^2 \\ \iff \left(8 - 2a^2\right)(x_n)^2 + (24 - 4ab)\,x_n y_n + \left(18 - 2b^2\right)(y_n)^2 &= 0 \end{aligned}$$

In order to have $(x_{n+1})^2 - 2(y_{n+1})^2 = (x_n)^2 - 2(y_n)^2$ we need

$$2a^2 = 8, \qquad 4ab = 24, \qquad 2b^2 = 18.$$

We further require that $a, b > 0$. We see that $a = 2$ and $b = 3$ solve all three equations.

(iii) Starting with $x_1 = 3$, $y_1 = 2$ we find:

$$\begin{aligned} x_1 &= 3, & y_1 &= 2; \\ x_2 &= 3 \times 3 + 4 \times 2 = 17, & y_2 &= 2 \times 3 + 3 \times 2 = 12; \\ x_3 &= 3 \times 17 + 4 \times 12 = 99, & y_3 &= 2 \times 17 + 3 \times 12 = 70. \end{aligned}$$

So $X = 99$ and $Y = 70$ is such a pair.

(iv) For the generated sequences, $(x_n), (y_n)$, we have

$$(x_n)^2 - 2(y_n)^2 = 1 \text{ for each } n.$$

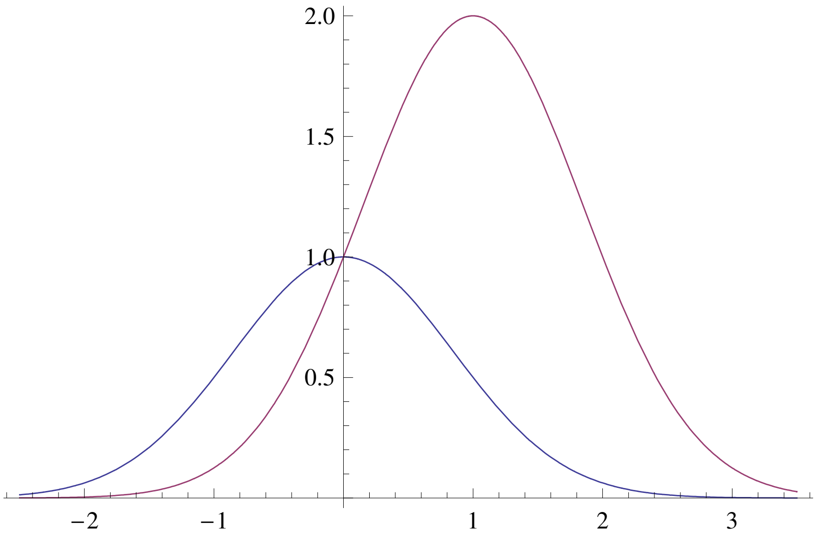
Also the integers $x_n$ and $y_n$ are getting increasingly larger because of how they are defined in (ii). So

$$\left(\frac{x_n}{y_n}\right)^2 - 2 = \frac{1}{(y_n)^2} \approx 0 \text{ for large } n,$$

and $x_n/y_n \approx \sqrt{2}$ as $x_n$ and $y_n$ are both positive.

**3.** (i) (A) is $-f(x)$; (B) is $f(-x)$; (C) is $f(x-1)$.

(ii) As $2^{2x-x^2} = 2 \times 2^{-(x-1)^2}$ then the graph of $y = 2^{2x-x^2}$ is the graph of $y = 2^{-x^2}$ translated to the right by 1 and stretched parallel to the $y$-axis by a factor of 2.




Plots of $y = 2^{-x^2}$ and $y = 2^{2x-x^2}$.

(ii) $c = \frac{1}{2}$. The graph of $2^{-(x-c)^2}$ is the graph of $2^{-x^2}$ translated $c$ to the right. The integral $I(c)$ represents the area under the graph between $0 \leqslant x \leqslant 1$. As the graph is symmetric/even and decreasing away from 0 then this area is maximised by having the apex half way along the interval $0 \leqslant x \leqslant 1$, i.e. at $x = 1/2$ which occurs when $c = \frac{1}{2}$.

**4.** (i) We can complete the squares in $x^2 - px + y^2 - qy = 0$ to get

$$\left(x - \frac{p}{2}\right)^2 + \left(y - \frac{q}{2}\right)^2 = \frac{p^2 + q^2}{4} \tag{1}$$

which is the equation of the circle with centre: $(p/2, q/2)$ and area: $\pi\left(p^2 + q^2\right)/4$. Either by checking the original question, or the rearranged one, we can see that

$$x^2 - px + y^2 - qy = \begin{cases} 0 & \text{at } (0,0)\,, \\ p^2 - p^2 + 0 = 0 & \text{at } (p,0)\,, \\ 0 + q^2 - q^2 = 0 & \text{at } (0,q)\,. \end{cases}$$

(ii) The area of $OPQ$ is $pq/2$. So

$$\frac{\text{area of circle } C}{\text{area of triangle } OPQ} = \left(\frac{\pi\left(p^2 + q^2\right)}{4}\right) \Big/ \left(\frac{1}{2}pq\right) = \frac{\pi\left(p^2 + q^2\right)}{2pq}.$$

Note

$$\frac{\pi\left(p^2 + q^2\right)}{2pq} \geqslant \pi \iff p^2 + q^2 \geqslant 2pq \iff (p - q)^2 \geqslant 0,$$

proving the required inequality.

(iii) Rearranging

$$\frac{\pi\left(p^2 + q^2\right)}{2pq} = 2\pi \iff p^2 + q^2 = 4pq \iff \left(\frac{p}{q}\right)^2 - 4\left(\frac{p}{q}\right) + 1 = 0,$$

which is a quadratic equation in $p/q$, and so

$$\frac{p}{q} = \frac{4 \pm \sqrt{16 - 4}}{2} = 2 \pm \sqrt{3}.$$

Now $p/q = \tan OQP$, $q/p = \tan OPQ$ and so

$$\{\tan OQP, \tan OPQ\} = \left\{2 - \sqrt{3}, 2 + \sqrt{3}\right\}$$

with the order depending on whether $p < q$ or $p > q$.

[It happens that $\arctan\left(2 - \sqrt{3}\right) = \pi/12$ and $\arctan\left(2 + \sqrt{3}\right) = 5\pi/12$, but appreciation of this was not expected.]

**5.** (i) After the first/second/third students have gone by the doors look like:

| Locker | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Student 1 | O | O | O | O | O | O | O | O | O | O | O | O | O | O |
| Student 2 | O | C | O | C | O | C | O | C | O | C | O | C | O | C |
| Student 3 | O | C | C | C | O | O | O | C | C | C | O | O | O | C |

We can see that the lockers now repeat in a pattern OCCCOO every 6 lockers. As $1000 = 166 \times 6 + 4$ we have 166 repeats of this pattern and 4 remaining lockers that go OCCC. So there are $166 \times 3 = 498$ closed lockers amongst the complete cycles and 3 further in the incomplete cycle. That is, there are 501 closed lockers in all.

(ii) After the fourth student has gone by we have the following:

| Locker | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Student 3 | O | C | C | C | O | O | O | C | C | C | O | O | O | C |
| Student 4 | O | C | C | O | O | O | O | O | C | C | O | C | O | C |

with the pattern repeating every 12 lockers in the form OCCOOOOOCCOC. Each cycle contains 5 closed and 7 open doors. Now $1000 = 83 \times 12 + 4$ and so we have $83 \times 5 = 415$ closed lockers amongst the complete cycles and 2 further amongst the incomplete cycle OCCO. In all then there are 417 closed lockers.

(iii) Locker 100 starts off closed (as all lockers do) and then its state is altered by every $n$th student where $n$ is a factor of 100, i.e. by students $1, 2, 4, 5, 10, 20, 25, 50, 100$. So 9 students change the state and as this is odd then overall the state will have been changed to open.

(iv) Locker 1000 starts off closed (as all lockers do) and then its state is altered by every $n$th student where $n$ divides 1000 and $n \leqslant 100$, i.e. by $1, 2, 4, 5, 8, 10, 20, 25, 40, 50, 100$. So 11 students change the door's state and as this is odd then overall the state will again have been changed to open.

**6.** (i) We have six possibilities:

A-B-C = St-L-Sw,      St-Sw-L,      L-St-Sw,      L-Sw-St,      Sw-L-St,      Sw-St-L.

The statement "I am the liar" cannot be made by St or L; this excludes the first four possibilities above.

The second statement "A is the liar" excludes Sw-St-L and so we are left with Sw-L-St. Answer: B is the Liar.

(The third statement is not actually needs but doesn't contradict the Sw-L-St arrangement.)

(ii) We have six possibilities:

P-Q-R = S-L-C,      S-C-L,      L-S-C,      L-C-S,      C-L-S,      C-S-L.

One of these statements is from a saint and so true. This means that the Liar has to follow the Saint in cyclic order and this means the only remaining possibilities are

P-Q-R = S-L-C,      L-C-S,      C-S-L.

In the first two cases the Contrarian follows the Liar and so tells the truth. But this contradicts the actual statements so the only possibility remaining is C-S-L. Answer: R is the Liar.

(iii) We have six possibilities:

X-Y-Z = S-L-C,      S-C-L,      L-S-C,      L-C-S,      C-L-S,      C-S-L.

We will take these case by case:

- S-L-C: As the Contrarian is following the Liar, statement 3 had to be true but isn't in this case.
- L-C-S: As the Contrarian is following the Liar, statement 2 had to be true but isn't in this case.
- C-S-L: As the Contrarian is following the Liar, statement 4 had to be true but isn't in this case.
- S-C-L: In this case, statement 4 is a lie and so the Contrarian would tell the truth in Statement 5 but doesn't.
- C-L-S: The Contrarian tells the truth to begin contradicting his nature.
- L-S-C: This is the only remaining case and is consistent.

Answer: X is the Liar.

**7.** (i) The empty word has zero length which is even. If a new word is formed by Rule 2 then $aWb$ will have the same parity of length as $W$ had. Also if $U$ and $V$ are even-length words then so will be $UV$. So new words formed from words of even length will themselves be even.

(ii)

Length 0 words: $\varnothing$.
Length 2 words: $ab$.
Length 4 words: $abab$, $aabb$
Length 6 words: $ababab$, $abaabb$, $aabbab$, $aababb$, $aaabbb$

(iii) In $\varnothing$ there are the same number of $a$s and $b$s, namely none. If $W$ has the same number then so will $aWb$, formed by Rule 2. Also if $U$ and $V$ each have the same number of $a$s and $b$s then so will $UV$. So new words formed by Rules 2 and 3 always have the same property.

(iv) A word of the form $aWbW'$ will be of length $2n+2$ if

$$\text{length}\,(W) + \text{length}\,(W') = 2n.$$

So if $W$ has length $2k \leqslant 2n$ then $W'$ has length $2\,(n-k)$. There are $C_k$ words of the former length and $C_{n-k}$ of the latter length. So we may generate $C_kC_{n-k}$ such words of length $2n+2$ in this manner for each $k$. That is,

$$\sum_{k=0}^{n} C_kC_{n-k}$$

in all. Further, because the uniqueness of form in the given hint, all words of length $2n+2$ are counted amongst these words and none are doubly counted. That is

$$C_{n+1} = \sum_{k=0}^{n} C_kC_{n-k}$$