Imperial College London

UNIVERSITY OF OXFORD

WARWICK
THE UNIVERSITY OF WARWICK

# MATHEMATICS ADMISSIONS TEST

**For candidates applying for Mathematics, Computer Science or one of their joint degrees at OXFORD UNIVERSITY and/or IMPERIAL COLLEGE LONDON and/or UNIVERSITY OF WARWICK**

**November 2022**

**Time Allowed: 2½ hours**

Please complete the following details in BLOCK CAPITALS. You must use a pen.

| | |
|---|---|
| Surname | |
| Other names | |

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|---|---|---|---|---|---|---|
| Candidate Number | **M** | | | | | |

This paper contains 7 questions of which you should attempt 5. There are directions throughout the paper as to which questions are appropriate for your course.

**A: Oxford Applicants:** if you are applying to Oxford for the degree course:

- Mathematics *or* Mathematics & Philosophy *or* Mathematics & Statistics, you should attempt Questions **1,2,3,4,5.**
- Mathematics & Computer Science, you should attempt Questions **1,2,3,5,6.**
- Computer Science *or* Computer Science & Philosophy, you should attempt **1,2,5,6,7.**

***Directions under A take priority over any directions in B which are relevant to you.***

**B: Imperial or Warwick Applicants:** if you are applying to the University of Warwick for Mathematics BSc, Master of Mathematics, or if you are applying to Imperial College for any of the Mathematics courses: Mathematics, Mathematics (Pure Mathematics), Mathematics with a Year Abroad, Mathematics with Applied Mathematics/Mathematical Physics, Mathematics with Mathematical Computation, Mathematics with Statistics, Mathematics with Statistics for Finance, you should attempt Questions **1,2,3,4,5.**

Further credit cannot be obtained by attempting extra questions. **Calculators are not permitted.**

Question 1 is a multiple choice question with ten parts. Marks are given solely for correct answers but any rough working should be shown in the space between parts. Answer Question 1 on the grid on Page 2. Each part is worth 4 marks.

Answers to questions 2-7 should be written in the space provided, continuing on to the blank pages at the end of this booklet if necessary. Each of Questions 2-7 is worth 15 marks.

| FOR OFFICE USE ONLY | Q1 | Q2 | Q3 | Q4 | Q5 | Q6 | Q7 |
|---|---|---|---|---|---|---|---|
| | | | | | | | |

**1.** For **ALL APPLICANTS.**

For each part of the question on pages 3-7 you will be given **five** possible answers, just one of which is correct. Indicate for each part **A-J** which answer (a), (b), (c), (d), or (e) you think is correct with a tick (✓) in the corresponding column in the table below. *Please show any rough working in the space provided between the parts.*

| | (a) | (b) | (c) | (d) | (e) |
|---|---|---|---|---|---|
| **A** | | | | | |
| **B** | | | | | |
| **C** | | | | | |
| **D** | | | | | |
| **E** | | | | | |
| **F** | | | | | |
| **G** | | | | | |
| **H** | | | | | |
| **I** | | | | | |
| **J** | | | | | |

**A.** How many real solutions $x$ are there to the equation $x|x| + 1 = 3|x|$?

(a) 0, (b) 1, (c) 2, (d) 3, (e) 4.

[Note that $|x|$ is equal to $x$ if $x \geqslant 0$, and equal to $-x$ otherwise.]

**B.** One hundred circles all share the same centre, and they are named $C_1$, $C_2$, $C_3$, and so on up to $C_{100}$. For each whole number $n$ between 1 and 99 inclusive, a tangent to circle $C_n$ crosses circle $C_{n+1}$ at two points that are separated by a distance of 2. Given that $C_1$ has radius 1, it follows that the radius of $C_{100}$ is

(a) 1, (b) 2, (c) $\sqrt{10}$, (d) 10, (e) 100.

Turn over

**C.** The equation $x^2 - 4kx + y^2 - 4y + 8 = k^3 - k$ is the equation of a circle

(a) for all real values of $k$.

(b) if and only if either $-4 < k < -1$ or $k > 1$.

(c) if and only if $k > 1$.

(d) if and only if $k < -1$.

(e) if and only if either $-1 < k < 0$ or $k > 1$.

**D.** A sequence has $a_0 = 3$, and then for $n \geqslant 1$ the sequence satisfies $a_n = 8\left(a_{n-1}\right)^4$. The value of $a_{10}$ is

(a) $\dfrac{2^{(2^{20})}}{3}$, (b) $\dfrac{6^{(2^{20})}}{3}$, (c) $\dfrac{3^{(2^{20})}}{2}$, (d) $\dfrac{18^{(2^{20})}}{2}$, (e) $\dfrac{6^{(2^{20})}}{2}$.

**E.** If the expression $\left(x + 1 + \dfrac{1}{x}\right)^4$ is fully expanded term-by-term and like terms are collected together, there is one term which is independent of $x$. The value of this term is

(a) 10, (b) 14, (c) 19, (d) 51, (e) 81.

**F.** Given that

$$\sin(5\theta) = 5\sin\theta - 20(\sin\theta)^3 + 16(\sin\theta)^5$$

for all real $\theta$, it follows that the value of $\sin(72^\circ)$ is

(a) $\sqrt{\dfrac{5+\sqrt{5}}{8}}$, (b) 0, (c) $-\sqrt{\dfrac{5+\sqrt{5}}{8}}$,

(d) $\sqrt{\dfrac{5-\sqrt{5}}{8}}$, (e) $-\sqrt{\dfrac{5-\sqrt{5}}{8}}$.

Turn over

**G.** For all real $n$, it is the case that $n^4+1=(n^2+\sqrt{2}n+1)(n^2-\sqrt{2}n+1)$. From this we may deduce that $n^4+4$ is

(a) never a prime number for any positive whole number $n$.

(b) a prime number for exactly one positive whole number $n$.

(c) a prime number for exactly two positive whole numbers $n$.

(d) a prime number for exactly three positive whole numbers $n$.

(e) a prime number for exactly four positive whole numbers $n$.

**H.** How many real solutions $x$ are there to the following equation?

$$\log_2(2x^3+7x^2+2x+3)=3\log_2(x+1)+1$$

(a) 0, (b) 1, (c) 2, (d) 3, (e) 4.

**I.** Alice and Bob each toss five fair coins (each coin lands on either heads or tails, with equal probability and with each outcome independent of each other). Alice wins if strictly more of her coins land on heads than Bob's coins do, and we call the probability of this event $p_1$. The game is a draw if the same number of coins land on heads for each of Alice and Bob, and we call the probability of this event $p_2$. Which of the following is correct?

(a) $p_1 = \dfrac{193}{512}$ and $p_2 = \dfrac{63}{256}$.

(b) $p_1 = \dfrac{201}{512}$ and $p_2 = \dfrac{55}{256}$.

(c) $p_1 = \dfrac{243}{512}$ and $p_2 = \dfrac{13}{256}$.

(d) $p_1 = \dfrac{247}{512}$ and $p_2 = \dfrac{9}{256}$.

(e) $p_1 = \dfrac{1}{3}$ and $p_2 = \dfrac{1}{3}$.

**J.** The real numbers $m$ and $c$ are such that the equation

$$x^2 + (mx + c)^2 = 1$$

has a repeated root $x$, and also the equation

$$(x - 3)^2 + (mx + c - 1)^2 = 1$$

has a repeated root $x$ (which is not necessarily the same value of $x$ as the root of the first equation). How many possibilities are there for the line $y = mx + c$?

(a) 0, (b) 1, (c) 2, (d) 3, (e) 4.

Turn over

**2.** For **ALL APPLICANTS.**

(i) Suppose $x$, $y$, and $z$ are whole numbers such that $x^2 - 19y^2 = z$. Show that for any such $x$, $y$ and $z$, it is true that

$$(x^2 + Ny^2)^2 - 19(2xy)^2 = z^2$$

where $N$ is a particular whole number which you should determine.

(ii) Find $z$ if $x = 13$ and $y = 3$. Hence find a pair of whole numbers $(x, y)$ with $x^2 - 19y^2 = 4$ and with $x > 2$.

(iii) Hence find a pair of positive whole numbers $(x, y)$ with $x^2 - 19y^2 = 1$ and with $x > 1$.

Is your solution the only such pair of positive whole numbers $(x, y)$? Justify your answer.

(iv) Prove that there are no whole number solutions $(x, y)$ to $x^2 - 25y^2 = 1$ with $x > 1$.

(v) Find a pair of positive whole numbers $(x, y)$ with $x^2 - 17y^2 = 1$ and with $x > 1$.

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**3.**

For **APPLICANTS IN** $\left\{\begin{array}{l}\textbf{MATHEMATICS}\\ \textbf{MATHEMATICS \& STATISTICS}\\ \textbf{MATHEMATICS \& PHILOSOPHY}\\ \textbf{MATHEMATICS \& COMPUTER SCIENCE}\end{array}\right\}$ **ONLY.**

*Computer Science* and *Computer Science & Philosophy* **applicants should turn to page 20.**

(i) Sketch $y = (x^2 - 1)^n$ for $n = 2$ and for $n = 3$ on the same axes, labelling any points that lie on both curves, or that lie on either the $x$-axis or the $y$-axis.

(ii) Without calculating the integral explicitly, explain why there is no positive value of $a$ such that $\displaystyle\int_0^a (x^2 - 1)^n \, dx = 0$ if $n$ is even.

If $n > 0$ is odd we will write $n = 2m - 1$ and define $a_m > 0$ to be the positive real number that satisfies

$$\int_0^{a_m} (x^2 - 1)^{2m-1} \, dx = 0,$$

if such a number exists.

(iii) Explain why such a number $a_m$ exists for each whole number $m \geqslant 1$.

(iv) Find $a_1$.

(v) Prove that $\sqrt{2} < a_2 < \sqrt{3}$.

(vi) Without calculating further integrals, find the approximate value of $a_m$ when $m$ is a very large positive whole number. You may use without proof the fact that $\displaystyle\int_0^{\sqrt{2}} (x^2 - 1)^{2m-1} \, dx < 0$ for any sufficiently large whole number $m$.

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If you require additional space please use the pages at the end of the booklet

**4.**

For **APPLICANTS IN** $\left\{\begin{array}{l}\textbf{MATHEMATICS}\\ \textbf{MATHEMATICS \& STATISTICS}\\ \textbf{MATHEMATICS \& PHILOSOPHY}\end{array}\right\}$ **ONLY.**

*Mathematics & Computer Science*, *Computer Science* and *Computer Science & Philosophy* **applicants should** **turn to page 20.**

(i) Sketch the graph of $y = \sqrt{x} - \dfrac{x}{4}$ for $x \geqslant 0$, and find the coordinates of the turning point.

(ii) Describe in words how the graph of $y = \sqrt{4x+1} - x - 1$ for $x \geqslant -\frac{1}{4}$ is related to the graph that you sketched in part (i). Write down the coordinates of the turning point of this new graph.

Point $A$ is at $(-1, 0)$ and point $B$ is at $(1, 0)$. Curve $C$ is defined to be all points $P$ that satisfy the equation $|AP| \times |BP| = 1$, that is; the distance from $P$ to $A$, multiplied by the distance from $P$ to $B$, is 1.

(iii) Find all points that lie on both the $x$-axis and also on the curve $C$.

(iv) Find an equation in the form $y = f(x)$ for the part of the curve $C$ in the region where $x > 0$ and $y > 0$. You should explicitly determine the function $f(x)$.

(v) Use part (ii) to determine the coordinates of any turning points of the curve $C$ in the region where $x > 0$ and $y > 0$.

(vi) Sketch the curve $C$, including any parts of the curve with $x < 0$ or $y < 0$ or both.

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If you require additional space please use the pages at the end of the booklet

**5.** For **ALL APPLICANTS.**

Alice is participating in a TV game show where $n$ *distinct* items are placed behind $n$ closed doors. The game proceeds as follows. Alice picks a door $i$ which is opened and the item behind it is revealed. Then the door is shut again and the host *secretly* swaps the item behind door $i$ with the item behind one of the neighbouring doors, $i-1$ or $i+1$. If Alice picks door 1, the host has to then swap the item with the one behind door 2; similarly, if Alice picks door $n$, the host has to swap the item with the one behind door $n-1$. Alice then gets to pick any door again, and the process repeats for a certain fixed number of rounds. At the end of the game, Alice wins all the items that were revealed to her.

As a concrete example, suppose $n = 3$, and if the original items behind the three doors were $(a_1, a_2, a_3)$, then if first Alice picks door 2, the arrangement after the host has swapped items could be either $(a_2, a_1, a_3)$ or $(a_1, a_3, a_2)$. So if Alice was allowed to pick twice, had she chosen door 2 followed by door 1, in the former case she would only get the item $a_2$, whereas in the latter she would get items $a_2$ and $a_1$. Alice's aim is to find a sequence of door choices that *guarantee* her winning a large number of items, no matter how the swaps were performed.

(i) For $n = 13$, give an *increasing* sequence of length 7 of *distinct* doors that Alice can pick that guarantees she wins 7 items.

(ii) For any $n$ of the form $2k+1$, give a strategy to pick an *increasing* sequence of $k+1$ *distinct* doors that Alice can use to guarantee that she wins $k+1$ items. Briefly justify your answer.

(iii) For $n = 13$, give a sequence of length 10 of doors that Alice can pick that guarantees she wins 10 items.

(iv) For any $n$ of the form $3k + 1$, give a strategy to pick a sequence of $2k + 2$ doors that Alice can use to guarantee that she wins $2k + 2$ items. Briefly justify your answer.

(v) (a) For $n = 3$, give a sequence of length 3 of doors that Alice can pick that guarantees she wins all 3 items.

(b) For $n = 5$, give a sequence of length 5 of doors that Alice can pick that guarantees she wins all 5 items.

(vi) For $n = 13$, give a sequence of length 11 of doors that Alice can pick that guarantees she wins 11 items.

(vii) For any $n$ of the form $4k + 1$, give a strategy to pick a sequence of $3k + 2$ doors that Alice can use to guarantee that she wins $3k + 2$ items. Briefly justify your answer.

(viii) For $n = 6$, is there a sequence of *any length* of doors that Alice can choose that will *guarantee* that she wins all 6 items? Justify your answer.

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If you require additional space please use the pages at the end of the booklet

**6.**

For **APPLICANTS IN** { **COMPUTER SCIENCE** / **MATHEMATICS & COMPUTER SCIENCE** / **COMPUTER SCIENCE & PHILOSOPHY** } **ONLY.**

This question is about *influencer networks*. An influencer network consists of $n$ *influencers* denoted by circles, and arrows between them. Throughout this question, each influencer holds one of two opinions, represented by either a $\triangle$ or a $\square$ in the circle. We say that an influencer $A$ follows influencer $B$ if there is an arrow from $B$ to $A$; this indicates that $B$ has ability to influence $A$.

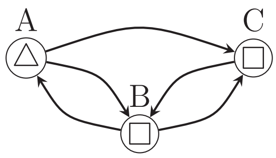



Figure 1

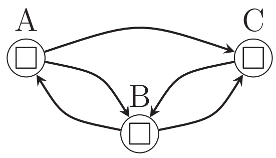
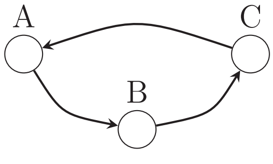



Figure 2

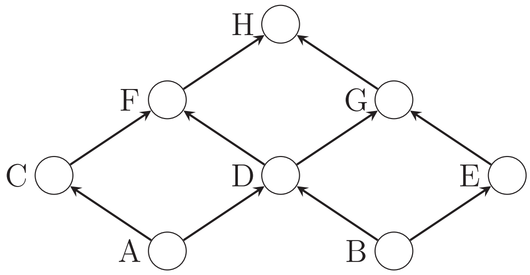
The example in Figure 1 above shows a network with $A$ and $B$ following each other, $B$ and $C$ following each other, and $C$ also following $A$. In this example, initially $B$ and $C$ have opinion $\square$, while $A$ has opinion $\triangle$. An influencer will change their mind according to the *strict* majority rule, that is, they change their opinion if *strictly* more than half of the influencers they are following have an opinion different from theirs. Opinions in an influence network change in rounds. In each round, each influencer will look at the influencers they are following and *simultaneously* change their opinion at the end of the round according to the strict majority rule. In the above network, after one round, $A$ changes their opinion because the only influencer they are following, $B$, has a differing opinion, and the network becomes as shown in Figure 2 above.

An influencer network with an initial set of opinions is *stable* if no influencer changes their opinion, and a network (with initial opinions) is *eventually stable* if after a finite number of rounds it becomes stable. The network in the above example is eventually stable as it becomes stable after one round.

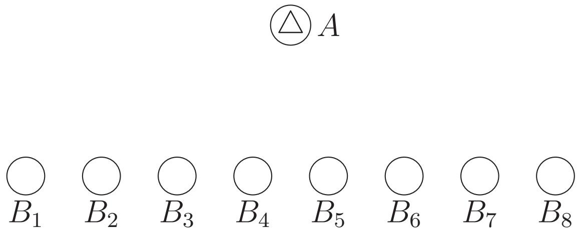
(i) A network of three influencers (without opinions) is shown below. Is this influencer network eventually stable regardless of the initial opinions of the influencers $A$, $B$ and $C$? Justify your answer.



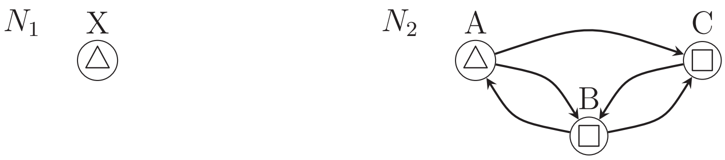

(ii) Another network of influencers (without opinions) is shown below. Is this influencer network eventually stable regardless of the initial opinions of the influencers? Justify your answer.




(iii) A *partial* network of influencers (without opinions for $B_1, \ldots, B_8$) is shown below. You can add *at most six* additional influencers, assign any opinion of your choice to the new influencers, and add any arrows to the network to describe follower relationships. Design a network that is eventually stable regardless of initial opinions, and has the property that when it becomes *stable* $A$ has opinion □ if and only if each of $B_1, B_2, \ldots, B_8$ had opinion □ at the start. Justify your answer.




(iv) You are given two influencer networks, $N_1$ and $N_2$, with disjoint sets of influencers shown below. Both are *eventually stable*. Suppose one of the influencers from network $N_2$ follows the influencer $X$ from the network $N_1$. Is the resulting network guaranteed to be eventually stable? Justify your answer.




(v) (a) Given a network with $n$ influencers, where the arrows are fixed, but you are allowed to assign *opinions* (△ or □) to each influencer, how many possible assignments of opinions is possible?

(b) Given an influencer network and an initial assignment of opinions, explain how you would determine whether the influencer network is eventually stable.

Justify your answer.

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If you require additional space please use the pages at the end of the booklet

**7.**

For **APPLICANTS IN** { **COMPUTER SCIENCE** / **COMPUTER SCIENCE & PHILOSOPHY** } **ONLY.**

A data operator is receiving tokens one by one through an *input* channel. The data operator will receive a total of $n$ tokens, where $n \geqslant 6$, and these are numbered as $1, 2, \ldots, n$. The data operator is required to pass these tokens to the output channel; however they must do so using a *valid* sequence. A *valid* sequence is one where all the odd tokens appear first, followed by the even tokens, or the other way round; furthermore all the odd tokens must appear in either increasing or decreasing order, and likewise all the even tokens must appear in increasing or decreasing order. All *valid* sequences for $n = 6$ are listed below.

| | | | |
|---|---|---|---|
| 1 3 5 2 4 6 | 1 3 5 6 4 2 | 5 3 1 2 4 6 | 5 3 1 6 4 2 |
| 2 4 6 1 3 5 | 2 4 6 5 3 1 | 6 4 2 1 3 5 | 6 4 2 5 3 1 |

The data operator has a storage unit that can hold a sequence, and can perform the following operations as they receive the tokens one by one.

- pass : Input token goes straight to the output channel.
- pop : Instead of using a token from the input, the token from the *right* end of the storage unit is removed (provided one exists) and sent to the output channel.
- pushL : Input token is pushed in at the left end of the storage unit.
- pushR : Input token is pushed in at the right end of the storage unit.

As an illustrative example, when $n = 6$, the storage and output channel are shown for the following sequence of operations, which results in the *valid* output sequence 1 3 5 6 4 2.

| Input Token | Operation | Storage | Output |
|---|---|---|---|
| 1 | pass | [ ] | 1 |
| 2 | pushR | [ 2 ] | 1 |
| 3 | pass | [ 2 ] | 1 3 |
| 4 | pushR | [ 2 4 ] | 1 3 |
| 5 | pass | [ 2 4 ] | 1 3 5 |
| 6 | pass | [ 2 4 ] | 1 3 5 6 |
| | pop | [ 2 ] | 1 3 5 6 4 |
| | pop | [ ] | 1 3 5 6 4 2 |

(i) For $n = 6$, which valid sequences can the data operator achieve?

(ii) For $n \geqslant 6$ and even, how many valid sequences are there? Justify your answer.

(iii) For $n \geqslant 6$ and even, how many valid sequences can be achieved by the data operator? Briefly justify your answer.

(iv) In the remainder of the question, $n \geqslant 9$ is a multiple of 3. A 3-*valid output sequence* is one where among the tokens $1, 2, \ldots, n$, all tokens of the form $3k$ appear together in increasing or decreasing order, likewise all tokens of the form $3k + 1$ appear together in increasing or decreasing order, and the same is the case for all tokens of the form $3k + 2$. As examples, the two sequences on the left below are 3-valid, whereas the two on the right are not – the first because it mixes groups and the second because although the groups are separate, the tokens of the form $3k+2$ are in neither increasing nor decreasing order.

| **3-valid** | **not 3-valid** |
|---|---|
| 1 4 7 9 6 3 8 5 2 | 1 3 5 6 4 2 7 8 9 |
| 2 5 8 1 4 7 3 6 9 | 2 8 5 9 6 3 1 4 7 |

For $n \geqslant 9$ and multiple of 3, how many 3-valid sequences of length $n$ are there? Justify your answer.

(v) For $n \geqslant 9$ and multiple of 3, given the input sequence of tokens $1, 2, \ldots, n$, how many 3-valid sequences can be achieved by the operator using a single storage unit and the operations `pass`, `pop`, `pushL` and `pushR`? Justify your answer.

End of last question

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